

On the structure of the set of self-similar quadruples of point vortices in the plane *

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1 Introduction and main results

A point vortex is a point in the plane with a non-zero real number, called its circulation, assigned to it. This is an idealization of a vortex thread in three-dimensional space, which has the shape of a straight line. Circulation defines the intensity and orientation of rotation of the thread itself, and consequently also of the surrounding space. A vortex with coordinates $(x, y) \in \mathbb{R}^2$ (or equivalently $z = x + iy \in \mathbb{C}$) and circulation κ forces the points in the plane to move, endowing a passive point $w \in \mathbb{C}$ with speed

$$i\kappa \frac{w - z}{|w - z|^2}.$$

Thus, the velocity vector at w is perpendicular to the relative position vector. Its length is inversely proportional to the distance between the points and directly proportional to circulation, while its orientation depends on the sign of circulation.

A finite system of point vortices in the plane evolves in such a way that the speed of each of the vortices is sum of the speed caused by the other vortices, in accordance with the above principle.

Evolution of vortex systems has been studied for over 130 years (see [1], [2], [5]). As early as 1883 Kirchhoff showed that this movement can be described using the formalism of Hamilton. Quite surprisingly, there are

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systems of three vortices which during evolution collapse to a point. This result was already included in the Groebli's work [4] from 1877. In 1979 Novikov [6] gave examples of systems of n vortices collapsing to a point for $n = 4$ and 5. His example (for $n = 4$) concerns in fact only one circulation (see Section 8 below) and the set of collapsing systems given by him (after a natural completion) can be identified with a circle in the plane. Therefore it constitutes a single smooth simple closed curve. In 1987, O'Neil [7] proved that collapsing systems exist for any n . His proof works for a variety of circulations. O'Neil was able to prove that at least some connected components of the set of self-similar systems are curves, although one cannot rule out isolated singularities or self-intersections. This is due to the fact that the methods used by him belong to algebraic geometry. They imply that the set is a one-dimensional algebraic variety.

The aim of this study is to take a closer look at the collection of self-similar systems for the circulation used by Novikov. It should be noted that O'Neil's results do not apply directly to this particular circulation. We introduce a numerical procedure searching for self-similar systems and we shall look at the set it finds. The resulting image suggests that for this circulation the set has five algebraic components, each of which is a smooth simple closed curve. The curves seem to have neither singularities nor self-intersections, although one of them seems to intersect two others in single points, so that the number of connected components could be three.

The set found by Novikov is one of these components. In this work we numerically find four other components. They differ from the Novikov component in that they are not flat circles in any two-dimensional affine space. In the final part of the work we refer to the results of O'Neil and point out that the existence of the components found here is not predicted by these results.

The results below are mainly numerical. The algebraic equation proposed in the work, describing the set of self-similar systems, is derived rigorously. However, any application of the equation to a given vortex problem gives a numerical solution subject to numerical error. Therefore the results of the paper suggesting that the set of solutions is a union of five smooth curves requires more rigorous mathematical reasoning. We put it down to another study.

2 Layout of the work

The next section contains basic definitions and facts, taken from the work of O'Neil. I give the definition of the dynamical system which is the sub-

ject of our interest. I recall the concept of self-similar systems of vortices and distinguish four specific situations, in which a system may be invariant, translational, rotational, or collapsing. I present some basic invariance properties of the problem, related to the action of the group of affine transformations in the plane. In particular this leads to the concept of configuration. Configuration space has dimension lower by four than that of the phase space.

In the following part of the work I propose to study the set of the self-similar configurations by representing it as a set of zeros of some vector field U (see definition 3.1). I prove that in fact the zeros of U coincide with the self-similar configurations. Next I propose a numerical procedure searching for self-similar configurations by seeking zeros of the field U with the gradient method applied to the function $|U|^2$. We want to apply this method to systems of four vortices. In order to test it, we formulate it directly for three vortices. We present the image of the set of self-similar points obtained with this method for three vortices and compare it with the well-known solution, which is a circle.

The next section refers to the Novikov examples of collapsing systems for $n = 4$. We check that the configurations described by him form a flat circle. Next I give the results of my numerical calculations. I present graphic images of plane curves obtained as projections to the coordinate system planes of the set of self-similar configurations for the circulation used by Novikov. The set of configurations was obtained numerically as the set of zeros of the vector field U . In these images one can easily see five connected components. One of them is isometric to a circle - that's the Novikov component. The other four are the main goal of this paper.

3 Basic definitions and facts

For a natural number n , a circulation is a sequence of non-zero real numbers $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbb{R}_*^n$, and a discrete system of n point vortices (briefly a *system*) is a point of the phase space

$$DV^{[n]} = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : \forall_{j \neq k} z_j \neq z_k\}.$$

According to the commonly accepted definition (see e.g. O'Neil [7, p. 384, def. 0.1], Aref [1]), we consider a dynamical system in the phase space $DV = DV^{[n]}$ defined by the vector field

$$V = V^{[n]} = (V_1, \dots, V_n), \quad V_j = (\sqrt{-1}) \sum_{k=1, k \neq j}^n \kappa_k \frac{z_j - z_k}{|z_j - z_k|^2}.$$

Therefore we are interested in the trajectories of the field, i.e., the curves $t \rightarrow z(t) \in DV^{[n]}$ satisfying

$$z'(t) = V^{[n]}(z(t)).$$

As we know from the elementary theory of dynamical systems, for any initial conditions $z(0) \in DV$ there exists exactly one such curve, defined in maximal domain being an interval, bounded or not.

We distinguish the following four types of vortex systems $Z \in DV$ depending on the value of $V = V^{[n]}(Z)$. (cf. O'Neil [7, def. 1.1.3, p. 387])

- Z is a *fixed point* (equilibrium) if $V_1 = \dots V_n = 0$.
- Z is a *translational* system if $V_1 = \dots V_n = v \neq 0$ for some $v \in C$.
- Z is *rotational* (relative equilibrium) if some constants $0 \neq \lambda \in R$, $z_0 \in C$, satisfy $\forall_l V_l = i\lambda(Z_l - z_0)$.
- Z is a *collapsing* system if for some constants $0 \neq \omega, z_0 \in C$ we have $\forall_l V_l = \omega(Z_l - z_0)$ and $Re(\omega) \neq 0$.

A system Z is called *self-similar* if, during evolution, it remains similar to the initial state. In other words, if $z(0) = Z, z'(t) = V(z(t))$, then for any t the system $z(t)$ can be obtained from Z by dilation composed with isometry. It is easy to prove (see e.g. O'Neil [7]), that Z is self-similar if and only if there is a constant $\omega \in C$, for which

$$\forall_{j,k} (V_k - V_j) = \omega(z_k - z_j),$$

and that in turn holds if and only if the system Z belongs to one of the above listed, mutually exclusive types.

Lemma 3.1. *If a system Z is collapsing in the sense of the above definition (i.e., $V_l(Z) = \omega(Z_l - z_0)$, $Re(\omega) \neq 0$), then it is self-similar and its temporal evolution is described by the formula*

$$z(t) = z_0 + \sqrt{2Re(\omega)t + 1} e^{i \frac{Im(\omega)}{2Re(\omega)} \ln(2Re(\omega)t + 1)} (Z - z_0).$$

For a rotational system, with $V_l = i\lambda(Z_l - z_0)$, we have

$$z(t) = z_0 + e^{i\lambda t} (Z - z_0).$$

Thus a rotational system rotates isometrically around z_0 at a constant speed, while for a collapsing system all distances between the points $z_k(t)$ tend to zero when $t \rightarrow t_0 = -\frac{1}{2Re(\omega)t}$.

Notice that $t_0 > 0$ if $Re(\omega) < 0$ and the system collapses to a point in finite time. If $Re(\omega) > 0$ then $t_0 < 0$. The system collapses to a point during negative (reversed) flow of time, while for the positive flow of time the system expands to infinity.

Proof. Due to the fact that the field V is translation invariant, we can address only Z such that

$$V(Z) = \omega Z.$$

I will show that there are real functions $r(t), \phi(t)$ satisfying $r(0) = 1, \phi(0) = 0$, for which the curve $z(t) = r(t)e^{i\phi(t)}Z$ is an integral curve. Due to the form of the field V we have

$$V(z(t)) = \frac{e^{i\phi(t)}}{r(t)}V(Z) = \omega \frac{e^{i\phi(t)}}{r(t)}Z.$$

We substitute this expression to the equations of motion $z'(t) = V(z(t))$ and we get

$$r'(t) + r(t)i\phi'(t) = \frac{\omega}{r(t)}.$$

It follows that

$$r'(t) = \frac{Re(\omega)}{r(t)}, \quad r^2(t)\phi'(t) = Im(\omega).$$

Of course, if $Re(\omega) = 0$, then $r(t)$ is constantly equal 1, and $\phi(t) = Im(\omega)t$. If $Re(\omega) \neq 0$, then the first equation gives

$$r^2(t) = 2Re(\omega)t + 1,$$

and from the second one we have $\phi'(t) = \frac{Im(\omega)}{2Re(\omega)t+1}$, so that

$$\phi(t) = \frac{Im(\omega)}{2Re(\omega)} \ln(2Re(\omega)t + 1).$$

□

There is a long tradition to consider self-similar systems. One of the arguments in favor of this is that for such systems we can explicitly solve the equations of motion. Another argument is as follows. One can consider systems which collapse in a more general sense, namely systems for which the distance of two or more vortices tends to zero in finite time during the evolution. In the paper published in 2007 (preprint 2004) Garduno and Lacombe [3] showed for $n = 3$ that every system collapsing in the broader sense is self-similar, therefore is collapsing in the sense adopted here. The question is still open for systems of four or more vortices.

I will now discuss a few properties of self-similar systems. This will facilitate presentation of results and make it more transparent. Let me start with the invariance with respect to similarities. The group H of the orientation preserving similarities of the plane (that is, linear transformations conserving angles, or, in other words, compositions of translations, rotations and dilations) is a four-dimensional connected, non-compact Lie group. Transformations belonging to H are invertible \mathbb{C} -linear maps

$$C \ni z \rightarrow az + b \in C, \quad a, b \in C, a \neq 0.$$

Note that this group acts freely and 2-transitively on \mathbb{C} . In other words, for any numbers $z_1, z_2, w_1, w_2 \in \mathbb{C}$ satisfying $z_1 \neq z_2$ and $w_1 \neq w_2$ there exists precisely one $\phi \in H$ such that $\phi(z_k) = w_k, k = 1, 2$. Thus, the natural action of H on $DV^{[n]}$ for $n \geq 2$ given by $\phi(z)_k = \phi(z_k)$ is free: if $\phi(z) = z$ for some z , then $\phi = I$. It follows that the orbits of the group H in the phase space are four dimensional submanifolds, form a foliation, and the quotient space $C^{[n]} = DV^{[n]}/H$, called the configuration space, is a manifold of dimension $2n - 4$. Next, the operation preserves the type of points in the sense of the above definition. For example, if Z is a collapse, i.e., it satisfies $V_k(Z) = \omega(Z_k - z_0)$, then $W = \phi(Z) = aZ + b$ is also a collapse, namely

$$\begin{aligned} V_k(W) &= V_k(aZ + b) = \frac{a}{|a|^2} V_k(Z) = \frac{a}{|a|^2} \omega(Z_k - z_0) = \\ &= \frac{\omega}{|a|^2} (aZ_k - az_0) = \frac{\omega}{|a|^2} (W_k - az_0 - b). \end{aligned}$$

Note that the constant ω has been rescaled by a positive coefficient $\frac{1}{|a|^2}$. Thus the phase is conserved. More generally, one can show, that if $W = \phi(Z) = aZ + b$ then the trajectories emanating from the points Z and W , respectively, are coupled by ϕ :

$$W(t) = \phi \left(Z \left(\frac{1}{|a|^2} t \right) \right).$$

It follows that it makes sense to speak of various types of configurations, and in particular of rotating or collapsing configurations.

A related kind of invariance is contained in the following: if $t \rightarrow z(t)$ is an integral curve for the circulation κ and $c \in \mathbb{R}_*$ then $t \rightarrow w(t) = z(ct)$ is an integral curve for the circulation $c\kappa$.

In the sequel we shall be interested (for a given circulation κ) in the set of self-similar collapsing and rotating configurations

$$S_\kappa = \{[Z] : \exists_{0 \neq \omega, z_0} V(Z) = \omega(Z - z_0)\}.$$

Here, $[Z] \in \mathbb{C}^{[n]}$ denotes the configuration corresponding to the system Z , i.e., the set of all systems similar to Z .

For natural n , let $DU^{[n]}$ denote the following open subset of \mathbb{C}^n

$$DU^{[n]} = \{w \in \mathbb{C}^n : \forall_k (w_k \neq 0, 1), \forall_{j \neq k} (w_k \neq w_j)\}.$$

Notice that each orbit of the group H acting on $DV^{[n]}$ contains precisely one point of the form $(w, 1, 0)$ and $w \in DU^{[n-2]}$. It follows that the configuration space for n -vortices can be naturally identified with the $(2n-4)$ -dimensional space $DU^{[n-2]}$.

For future use, let us introduce the following vector field

Definition 3.1. For natural n and for $\kappa \in \mathbb{R}_*^{n+2}$, we define on the set $DU^{[n]}$ the vector field $U = U^{[n]}$ by the formula

$$U_k(w) = (\kappa_k + \kappa_{n+2}) \frac{w_k}{|w_k|^2} + \kappa_{n+1} \left(1 + \frac{w_k - 1}{|w_k - 1|^2} \right) - w_k \kappa_k \left(\frac{1 - w_k}{|1 - w_k|^2} + \frac{w_k}{|w_k|^2} \right) - w_k (\kappa_{n+1} + \kappa_{n+2}) + \sum_{j \leq n, j \neq k} \kappa_j \left(\frac{w_k - w_j}{|w_k - w_j|^2} + \frac{w_j}{|w_j|^2} - w_k \left(\frac{1 - w_j}{|1 - w_j|^2} + \frac{w_j}{|w_j|^2} \right) \right).$$

4 Self-similar systems as zeros of a vector field

In this section I will prove that the task of determining all self-similar systems is equivalent to determination of all zeros of the vector field U introduced above.

For the curve $z(t)$ in the space $DV^{[n]}$ we consider the family of affine transformations

$$\mathbb{C} \ni z \rightarrow \phi_t(z) = \frac{z - z_n(t)}{z_{n-1}(t) - z_n(t)} \in \mathbb{C}.$$

If each point $z(t)$ is transformed by the similarity ϕ_t (depending on t), we get a curve, whose last two coordinates are always equal to 1 and 0. It is therefore a curve of the form $(w(t), 1, 0)$, where

$$w_k(t) = \phi_t(z_k(t)) = \frac{z_k(t) - z_n(t)}{z_{n-1}(t) - z_n(t)}$$

for $k \leq n-2$. The curve $w(t)$ has values in $DU^{[n-2]}$.

Let $\kappa = (\kappa_j)_{j=1..n}$ be a fixed circulation and $U = U^{[n-2]}$ be the vector field on $DU^{[n-2]}$ defined above.

Theorem 4.1. *If $z(t)$ is an integral curve of the field $V^{[n]}$ then the curve $t \rightarrow w(t) \in DU^{[n-2]}$ satisfies the equation*

$$w'_k(t) = \frac{iU(w(t))}{|z_{n-1}(t) - z_n(t)|^2}.$$

Proof. Denote $z_{jk} = z_j - z_k$ and $\zeta = z_{n-1} - z_n$. Since $z(t)$ is an integral curve of V ,

$$z'_k = i \sum_{j \neq k} \kappa_j \frac{z_{kj}}{|z_{kj}|^2}.$$

Hence

$$-iz'_{kp} = (\kappa_k + \kappa_p) \frac{z_{kp}}{|z_{kp}|^2} + \sum_{j \neq k, p} \kappa_j \left(\frac{z_{kj}}{|z_{kj}|^2} + \frac{z_{jp}}{|z_{jp}|^2} \right).$$

Furthermore $w_k = z_{kn}/\zeta$, so that

$$\begin{aligned} -i\zeta^2 w'_k &= -i\zeta^2 \frac{z'_{kn}\zeta - z_{kn}\zeta'}{\zeta^2} = -iz'_{kn}\zeta - z_{kn}(-i\zeta') = \\ &= \left((\kappa_k + \kappa_n) \frac{z_{kn}}{|z_{kn}|^2} + \sum_{j \neq k, n} \kappa_j \left(\frac{z_{kj}}{|z_{kj}|^2} + \frac{z_{jn}}{|z_{jn}|^2} \right) \right) \zeta + \\ &- z_{kn} \left((\kappa_{n-1} + \kappa_n) \frac{\zeta}{|\zeta|^2} + \sum_{j \leq n-2} \kappa_j \left(\frac{z_{n-1,j}}{|z_{n-1,j}|^2} + \frac{z_{jn}}{|z_{jn}|^2} \right) \right). \end{aligned}$$

Npw we use

$$z_{kn} = w_k \zeta, \quad z_{kj} = z_{kn} - z_{jn} = (w_k - w_j) \zeta.$$

Therefore

$$\begin{aligned} -i|\zeta|^2 w'_k &= (\kappa_k + \kappa_n) \frac{w_k - w_n}{|w_k - w_n|^2} + \sum_{j \neq k, n} \kappa_j \left(\frac{w_k - w_j}{|w_k - w_j|^2} + \frac{w_j - w_n}{|w_j - w_n|^2} \right) + \\ &- (w_k - w_n) \left(\kappa_{n-1} + \kappa_n + \sum_{j \leq n-2} \kappa_j \left(\frac{w_{n-1} - w_j}{|w_{n-1} - w_j|^2} + \frac{w_j - w_n}{|w_j - w_n|^2} \right) \right) = \\ &= (\kappa_k + \kappa_n) \frac{w_k}{|w_k|^2} + \kappa_{n-1} \left(1 + \frac{w_k - 1}{|w_k - 1|^2} \right) - w_k \kappa_k \left(\frac{1 - w_k}{|1 - w_k|^2} + \frac{w_k}{|w_k|^2} \right) - \\ &- w_k (\kappa_{n-1} + \kappa_n) + \sum_{j \leq n-2, j \neq k} \kappa_j \left(\frac{w_k - w_j}{|w_k - w_j|^2} + \frac{w_j}{|w_j|^2} - w_k \left(\frac{1 - w_j}{|1 - w_j|^2} + \frac{w_j}{|w_j|^2} \right) \right). \end{aligned}$$

□

As I mentioned previously, if a self-similar system is transformed by a similarity, we get a self-similar system again. In other words, self-similarity is a characteristic property and can be tested in the configuration space. It follows that in order to determine all self-similar systems, it is enough to determine those self-similar systems z , for which $z_{n-1} = 1$, $z_n = 0$. Each of the self-similar systems found will give a four space of self-similar systems, namely the orbit under the action of the group H .

In the remaining part of this section I shall therefore deal only with self-similar systems $z \in DV^{[n]}$ of the form $z = (w, 1, 0)$, $w \in DU^{[n-2]}$.

Theorem 4.2. *For $w = (w_1, \dots, w_{n-2}) \in DU^{[n-2]}$, a system of n vortices $z = (w, 1, 0) \in DV^{[n]}$ is self-similar if and only if w is a zero of the vector field U defined in the previous section.*

Proof. The trajectory $z(t)$ of the field V emerging from $z = (w, 1, 0)$ is self-similar if and only if the curve $(w(t), 1, 0) \in C^n$, defined in the proof above, is self-similar. However, any two systems of the form $(w(t), 1, 0)$ have two coordinates equal, so they are similar only if the trajectory $w(t) \in C^{n-2}$ is constant. According to the formula from the Theorem, this occurs when w is a zero of the field U , because $z_n(t) - z_{n-1}(t) \neq 0$. \square

5 Systems of three vortices

The above theorem will be applied for $n = 4$, but for control purposes we apply it to $n = 3$, where the result is well known.

As we know, self-similar collapsing systems may exist only for circulations κ , for which $L = \sum_{l < j} \kappa_l \kappa_j = 0$. Taking homogeneity into account, we can consider only circulations with $\kappa_1 = -1$. For the circulation $\kappa = (-1, \kappa_2, \kappa_3)$ the condition $L = 0$ gives

$$\kappa_2 \kappa_3 = \kappa_2 + \kappa_3.$$

Therefore, circulations under consideration depend only on one parameter λ and one can take

$$\kappa_1 = -1, \quad \kappa_2 = \lambda, \quad \kappa_3 = \frac{\lambda}{\lambda - 1}.$$

Our claim is that the point $(w, 1, 0)$ is self-similar if and only if it nullifies the expression

$$U(w) = (\kappa_1 + \kappa_3) \frac{w}{|w|^2} + \kappa_2 \left(1 + \frac{w - 1}{|w - 1|^2} \right) - w \kappa_1 \left(\frac{1 - w}{|1 - w|^2} + \frac{w}{|w|^2} \right) - w(\kappa_2 + \kappa_3) =$$

$$= \frac{1}{\lambda - 1} \frac{w}{|w|^2} + \lambda \left(1 + \frac{w - 1}{|w - 1|^2} \right) + w \left(\frac{1 - w}{|1 - w|^2} + \frac{w}{|w|^2} \right) - w \frac{\lambda^2}{\lambda - 1}.$$

Multiplying both sides by $(\lambda - 1)\bar{w}(\bar{w} - 1)$ we get

$$\bar{w} - 1 + \lambda(\lambda - 1)\bar{w}^2 - (\lambda - 1)w - w\bar{w}(\bar{w} - 1)\lambda^2 = 0.$$

Now we substitute $w = x + yi$:

$$x - yi - 1 + \lambda(\lambda - 1)(x^2 - y^2 - 2xyi) - (\lambda - 1)(x + yi) - (x^2 + y^2)(x - yi - 1)\lambda^2 = 0$$

and take the imaginary part:

$$-y - 2\lambda(\lambda - 1)xy - (\lambda - 1)y + (x^2 + y^2)y\lambda^2 = 0.$$

After dividing by $y\lambda^2$, this gives

$$(x^2 + y^2) - 2x\frac{\lambda - 1}{\lambda} - \frac{1}{\lambda} = \left(x - \frac{\lambda - 1}{\lambda} \right)^2 + y^2 - \left(\frac{\lambda - 1}{\lambda} \right)^2 - \frac{1}{\lambda} = 0.$$

Therefore

$$\left(x - \frac{\lambda - 1}{\lambda} \right)^2 + y^2 = \left(\frac{\lambda - 1}{\lambda} \right)^2 + \frac{1}{\lambda} = \frac{\lambda^2 - \lambda + 1}{\lambda^2}.$$

This is an equation of a circle, since the determinant of the quadratic function $\lambda^2 - \lambda + 1$ is negative.

Similarly, the real part

$$x - 1 + \lambda(\lambda - 1)(x^2 - y^2) - (\lambda - 1)x - (x^2 + y^2)(x - 1)\lambda^2$$

can be decomposed

$$(\lambda x^2 - 2(\lambda - 1)x + \lambda y^2 - 1)(1 - \lambda x).$$

Finally, the solution set consists of the point $(\frac{1}{\lambda}, 0)$ and a circle centered at $(\frac{\lambda - 1}{\lambda}, 0)$ with radius $\sqrt{\frac{\lambda^2 - \lambda + 1}{\lambda^2}}$. This is consistent with the formulas in Aref's work [1]. He says that if two vortices are placed at the points

$$\left(\frac{\kappa_3}{\kappa_3 + \kappa_2}, 0 \right), \quad \left(-\frac{\kappa_2}{\kappa_3 + \kappa_2}, 0 \right),$$

then the third one is in the circle with center $(0, 0)$ and radius $\sqrt{\frac{\kappa_1 + \kappa_2 + \kappa_3}{\kappa_2 + \kappa_3}}$. Substitution of our circulations gives the vortices positions

$$\left(\frac{1}{\lambda}, 0 \right), \quad \left(-\frac{\lambda - 1}{\lambda}, 0 \right),$$

and the radius

$$\sqrt{\frac{\lambda^2 - \lambda + 1}{\lambda^2}}.$$

It turns out that the radii are the same, but the two vortex positions obtained by Aref as well as the center of the circle are translated with respect to ours by a vector $[1 - \frac{1}{\lambda}, 0]$.

It is well known that the circle contains both collapsing and rotating points, while the isolated point is a fixed point.

6 A numerical procedure

It is clear from Theorem 4.2 that in order to find self-similar rotating and collapsing systems of n vortices, we have to find zeros of the vector field U defined in the space $DU^{[n-2]}$ of dimension $2n - 4$ by Definition 3.1. We shall use a simple numerical approach.

Zeros of the vector field U are zeros (and hence the absolute minima) of the non-negative smooth function

$$DU \ni w \rightarrow f(w) = |U(w)|^2 \in \mathbb{R}^+.$$

We search the minima with a simple gradient method. The parameters of the procedure are the starting point P , the initial time step dt , the desired accuracy ϵ , and the maximum number of steps N . The procedure calculates repeatedly (in a loop) the gradient at the current point p , the shifted point $q = p - \nabla f dt$, and the value of the function at q . This value is compared with the value at the point p . If the new value is smaller, then q becomes the current point. If not, then we halve the step. We stop when the value of the function falls below a predetermined error ϵ or after the maximum number of steps N has been reached. If, after the shutdown of the procedure, the function value is not less than ϵ , we believe that the point we found is a minimum which is not an absolute minimum (i.e., not a zero of the vector field U) and therefore the point is discarded.

The gradient can be calculated by approximating the derivative by the difference ratio

$$\frac{df}{dx} \approx \frac{f(x + dx) - f(x)}{dx}$$

for a small increment dx , but if the function is a polynomial (or, more generally, is analytic) it is better to use the derivative calculated analytically. This is particularly important when we are near the minimum point.

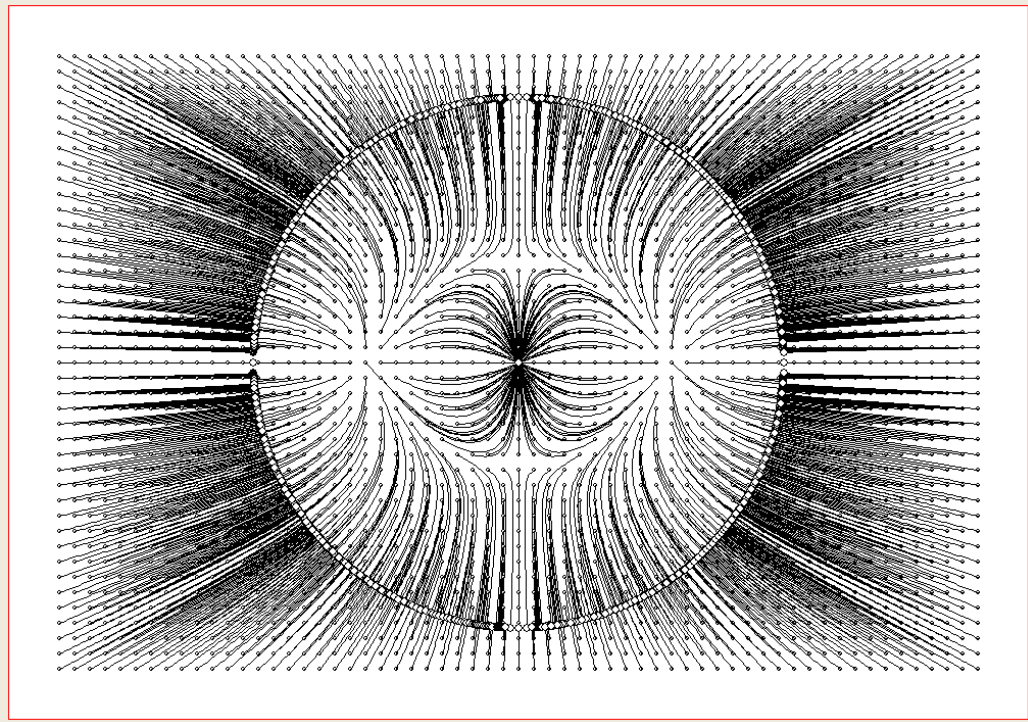
The procedure is applied as follows. In some preferred region of DU (e.g., in a box), we choose a large number of random starting points. For each of

them the gradient procedure is executed. The absolute minima found form a discrete collection, whose graphic image is supposed to provide information on the properties of the set of self-similar rotating and collapsing systems.

7 Graphic images of systems for $n=3$

For $n = 3$ the set of self-similar systems is a well known circle, as discussed above in Example 5. The set can be obtained numerically by searching zeros of the vector field indicated above.

Figure 1: Gradient lines of $|U(w)|^2$ convergent to the self-similar collapsing configurations of three vortices. The gradient is calculated numerically.



The calculations presented graphically in Fig. 1 are performed for the circulation $\kappa = (-1, 2, 2)$. According to [1] (as well as to the calculations from Section 5), if two vortices are placed at the points $(0, 0)$ and $(1, 0)$, then the third one point in the circle with center $(1/2, 0)$ and radius $\sqrt{3}/2 \approx 0.866$. Thus, it intersects the Ox axis at points $(-0.366, 0)$ and $(1.386, 0)$.

Fig. 1 contains the image of the gradient lines of the function $|U(w)|^2$ in this case. The gradient was calculated as the difference ratio with $dx = 10^{-6}$. Starting points for the gradient procedure (indicated in the figure with small

circles) are selected from the rectangle $[-1, 2] \times [-1, 1]$ in a regular grid of rectangular step 0.05. Gradient lines originating at these points converge to the minima (slightly larger circles). Except for the fixed point $(0.5, 0)$, they are rotating and collapsing systems lying on the said circle.

Figure 2: Gradient lines of $|U(w)|^2|w|^2|w-1|^2$ convergent to the self-similar collapsing configurations of three vortices. The gradient is calculated analytically.

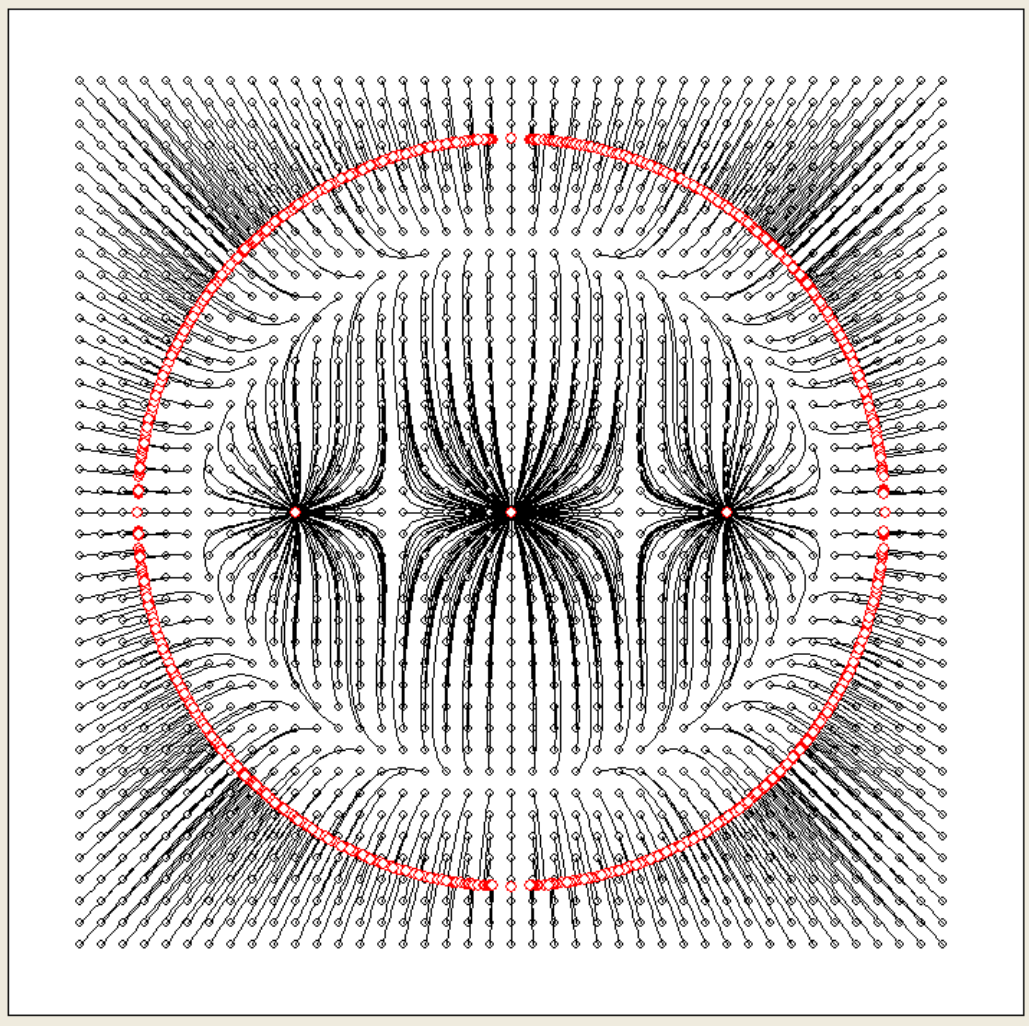


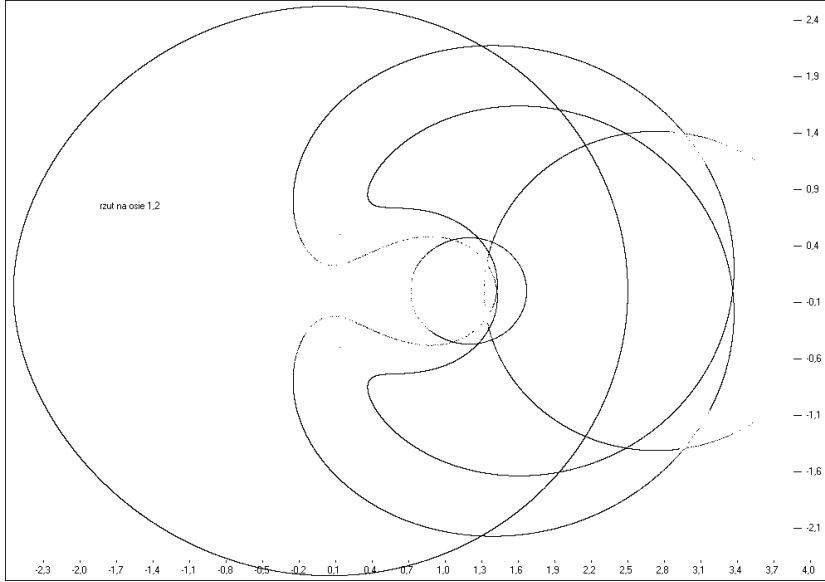
Fig. 2 shows a similar picture for the same circulation, step, and data grid. The main difference lies in the fact that the minimum was counted for the function $|U|^2$ multiplied by $|w|^2|w-1|^2$. Therefore, the graph contains two false minima at $(0,0)$ and $(1,0)$. Other minima (absolute) are unchanged because multiplying the studied function by a positive function

does not change its zeros. In this representation, the minimized function is a polynomial and the gradient was calculated from an analytical formula.

8 Novikov examples for $n=4$

In 1979, Novikov gave in [6] examples of collapsing systems for $n = 4$. In these examples, four points must be the vertices of a parallelogram. Vorticity of opposite corners must be equal.

Figure 3: Projection of the set of self-similar configurations of four vortices to the 1-2 plane.



Thanks to the invariance under the action of similarities and the homogeneity of vorticity, we can assume that the points and their vorticity are as follows

$$(w + 1, w, 1, 0) \in \mathbb{C}^4, \quad (1, \lambda, \lambda, 1) \in \mathbb{R}_*^4.$$

The necessary condition for the existence of collapsing systems

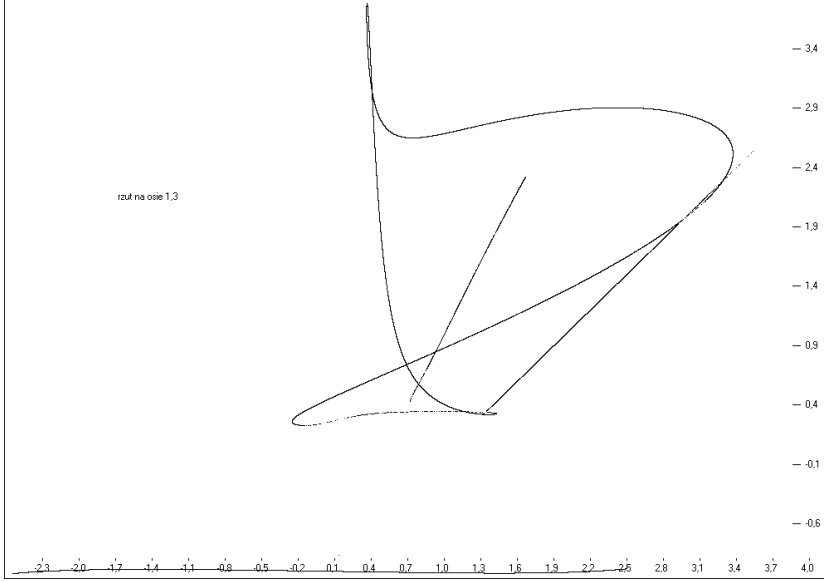
$$L = \sum \kappa_i \kappa_j = \lambda^2 + 4\lambda + 1 = 0$$

has to be satisfied. Therefore λ can take on one of two values $\lambda_{\pm} = -(2 \pm \sqrt{3})$.

According to Novikov, the sum of products of squared lengths of the diagonals multiplied by the intensity of any end-point of the diagonal must be zero. Here, this means that

$$1 \cdot |(w + 1) - 0|^2 + \lambda \cdot |w - 1|^2 = 0.$$

Figure 4: Projection of the set of self-similar configurations of four vortices to the 1-3 plane.



Thus the ratio of the length of the diagonal squares should be $-\lambda$:

$$\frac{|w+1|^2}{|w-1|^2} = -\lambda.$$

For $\lambda = \lambda_+ = -(2 + \sqrt{3})$ this means that

$$\frac{|w+1|^2}{|w-1|^2} = 2 + \sqrt{3}.$$

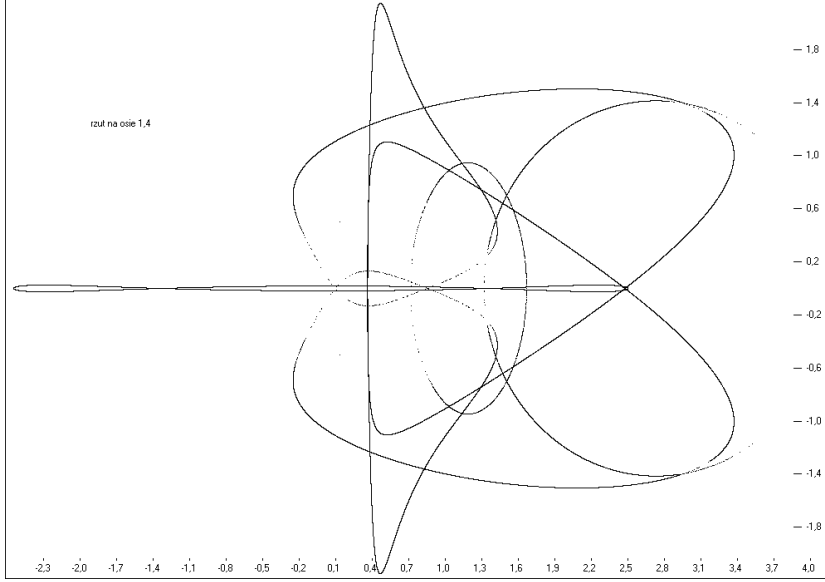
Denoting $w = x + iy$ we have

$$\begin{aligned} (2 + \sqrt{3})((x-1)^2 + y^2) &= (x+1)^2 + y^2, \\ (1 + \sqrt{3})(x^2 + y^2 + 1) &= 2(3 + \sqrt{3})x, \\ x^2 + y^2 + 1 &= 2 \frac{3 + \sqrt{3}}{1 + \sqrt{3}} x = 2 \frac{(3 + \sqrt{3})(\sqrt{3} - 1)}{2} x = 2\sqrt{3}x, \\ (x - \sqrt{3})^2 + y^2 &= 2. \end{aligned}$$

The set of solutions in the four-dimensional configuration space is isometrically an ellipse lying in the plane $w_1 = w_2 + 1$. Its projection onto each of the two complex axes Ow_1 , Ow_2 , is a circle. This ellipse can be described parametrically by

$$\theta \rightarrow (1 + \sqrt{3} + \sqrt{2}e^{i\theta}, \sqrt{3} + \sqrt{2}e^{i\theta}).$$

Figure 5: Projection of the set of self-similar configurations of four vortices to the 1-4 plane.



Let us look at the set of self-similar configurations for the second value, i.e., for $\lambda = \lambda_- = -(2 - \sqrt{3})$. The condition

$$\frac{|w+1|^2}{|w-1|^2} = 2 - \sqrt{3}$$

is equivalent to

$$2 + \sqrt{3} = \frac{|w-1|^2}{|w+1|^2} = \frac{|(-w)+1|^2}{|(-w)-1|^2}.$$

Thus, the set of solutions is a circle homothetic to the previous one.

One can look at this relationship as follows. Multiplying a circulation by a constant does not change the set of self-similar configurations, and moreover $\lambda_+ \lambda_- = 1$, therefore the following circulations are equivalent (they have the same set of self-similar configurations):

$$(1, \lambda_-, \lambda_-, 1) = (1, \lambda_+^{-1}, \lambda_+^{-1}, 1) \approx (\lambda_+, 1, 1, \lambda_+).$$

Permutation group of four elements acts on vortex systems (positions and circulations) and conserves their dynamics. In order to obtain self-similar configurations for the circulation $(1, \lambda_+, \lambda_+, 1)$ one can treat the system with a permutation replacing the first two coordinates and at the same time the last two coordinates. Then the system $(w+1, w, 1, 0)$ will transform into $(w, w+1, 0, 1)$. It can be represented by a system having 1 and 0 on the

last two coordinates if we act with a similarity $z \rightarrow 1 - z$. We receive $(-w + 1, -w, 1, 0)$, which shows that indeed the circle transforms into a homothetic circle.

The above remarks about symmetry occurring here can be further developed. Circulations of the form $(1, \lambda, \lambda, 1)$ admit a four-element invariance group $G = Z_2 \times Z_2$. It acts on self-similar configurations in such a way that one generator, say $\phi \in G$, swaps the first and last coordinate of the system, while the second generator — $\psi \in G$ — swaps the other two coordinates. In order to identify this action on the Novikov circle we take $u = (w + 1, w, 1, 0)$ and replace the images $\phi(u) = (0, w, 1, w + 1)$ and $\psi(u) = (w + 1, 1, w, 0)$ with similar systems, having 1 and 0 on the last coordinates. The needed similarities are $z \rightarrow \alpha(z) = 1 - \frac{z-1}{w}$ and $z \rightarrow \beta(z) = \frac{z}{w}$. Thus

$$[\phi(u)] = [\alpha(\phi(u))] = \left[\left(\frac{1}{w} + 1, \frac{1}{w}, 1, 0 \right) \right],$$

and

$$[\psi(u)] = [\beta(\psi(u))] = \left[\left(\frac{1}{w} + 1, \frac{1}{w}, 1, 0 \right) \right].$$

It turns out that in terms of w the operation of each of the selected generators of the group G on the Novikov circle is the map $w \rightarrow 1/w$. This is a composition of the symmetry in the Ox axis (complex conjugation $w \rightarrow \bar{w}$) and the inversion $w \rightarrow w/|w|^2$ with respect to the circle with center $(0, 0)$ and radius 1. In fact, the inversion preserves each of the two circles. That's because inversions map circles to circles, and it is clear that the points of intersection of the Novikov circle with the Ox axis, i.e., points $(\sqrt{3} - \sqrt{2}, 0)$ and $(\sqrt{3} + \sqrt{2}, 0)$ are conjugated, because $(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2}) = 1$. In the sequel, it will be interesting to see if other components of the self-similar configurations set for the Novikov circulation are also preserved by the group G .

Let us also note that the examples of Novikov concern actually only one (up to permutation and scaling) circulation $(1, \lambda, \lambda, 1)$. This follows from the fact that scaling the circulation does not change the location of collapsing systems. Permuting the circulation coordinates corresponds to permuting the coordinates of the self-similar system.

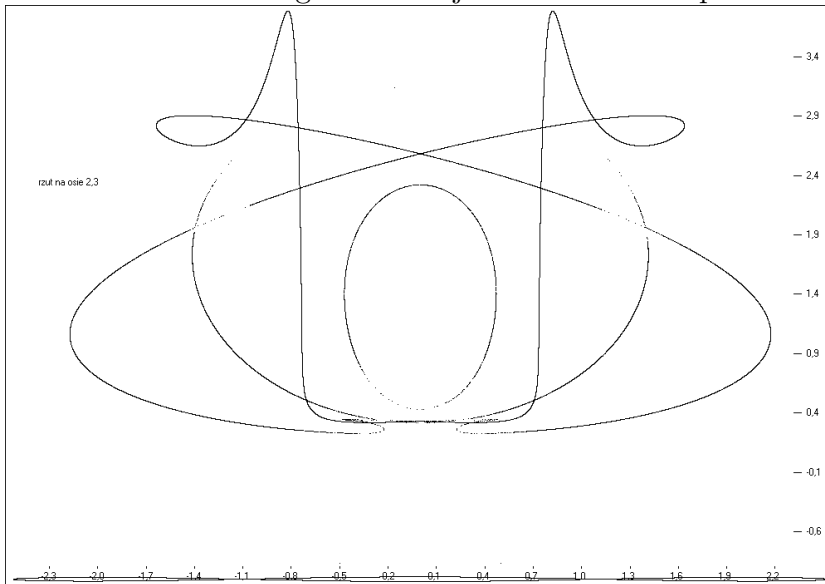
Novikov circulation has two positive and two negative entries. O'Neil's theorem proves the existence of collapses only when one of the configurations is linear rotating, and this fact is proved by O'Neil for circulations κ , for which one of the entries κ_l has sign opposite to the signs of all remaining entries. This is not satisfied by the Novikov circulation. Thus the existence of collapses for the Novikov circulation does not follow from the O'Neil The-

orem. In the next section we shall show that for this circulation, components other than the one found by Novikov do exist.

9 New examples of self-similar systems

The numerical methods proposed above for hunting self-similar rotating and collapsing systems will be applied now to producing images of the set of such systems for the circulation used by Novikov in his examples. My calculations suggest that for this circulation, there are other connected components of the set of self-similar collapsing and rotating systems, different from the ellipse found by Novikov. The calculations were performed by searching zeros of the vector field U as in Def. 3.1.

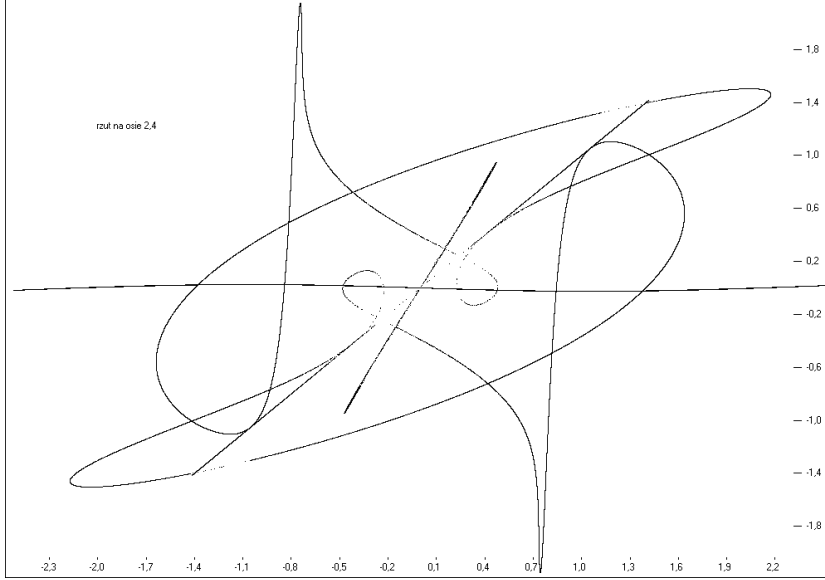
Figure 6: Projection to the 2-3 plane.



Graphic images in Figures 3 to 8 were obtained as described in section 6. In the four-dimensional cube $[-3.0, +3.0]^4$ we generate random points and then, using the gradient method, we lead gradient curves converging to a minimum of the function $|U(w)|^2$. If the value found is less than 10^{-8} , we believe we found a zero of the vector field U , therefore, a self-similar configuration. The resulting set of systems is projected to the plane coordinates for further analysis.

I present the views of the set S of self-similar configurations projected to various planes of the coordinate system. The Novikov circle consists of

Figure 7: Projection to the 2-4 plane.



points of $(1 + w, w) \in \mathbb{C}^2$, with $w \in \mathbb{C}$ located on the circle $|w - \sqrt{3}|^2 = 2$. This set can be represented parametrically

$$\theta \rightarrow (1 + \sqrt{3} + \sqrt{2} \cos \theta, \sqrt{2} \sin \theta, \sqrt{3} + \sqrt{2} \cos \theta, \sqrt{2} \sin \theta).$$

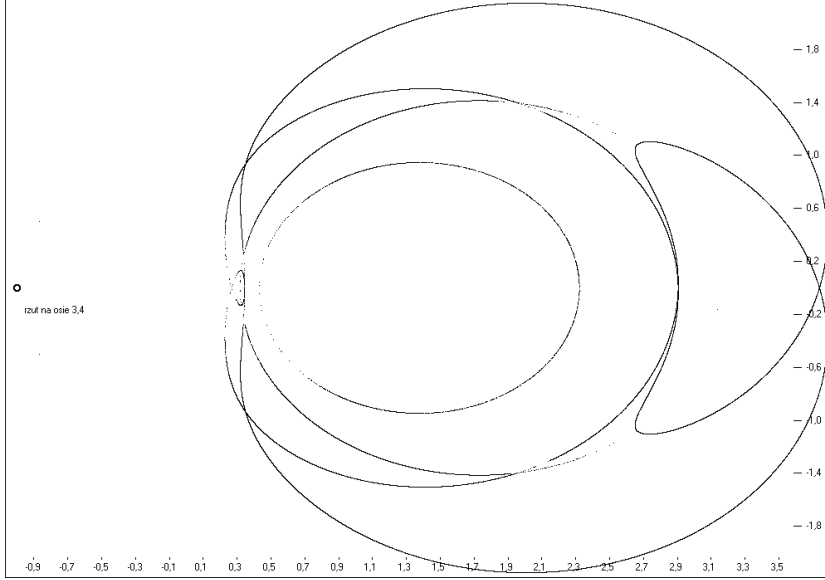
Thus, the projections to the planes of the coordinate system will be circles or segments. In order to easily identify the projection in the graphic image, below I list approximate coordinates of the end-points, if the projection image is a segment, and the range of the coordinates, if the projection image is a circle.

- π_{12} : circle , $x_1 \in [+1.31784, +4.14626], x_2 \in [-1.41421, +1.41421]$
- π_{13} : segment , $L = [+1.31784, +0.317837], P = [+4.14626, +3.14626]$
- π_{14} : circle , $x_1 \in [+1.31784, +4.14626], x_4 \in [-1.41421, +1.41421]$
- π_{23} : circle , $x_2 \in [-1.41421, +1.41421], x_3 \in [+0.317837, +3.14626]$
- π_{24} : segment , $L = [-1.41421, -1.41421], P = [+1.41421, +1.41421]$
- π_{34} : circle , $x_3 \in [+0.317837, +3.14626], x_4 \in [-1.41421, +1.41421]$

Here π_{kl} means the coordinate system plane spanned by the axes $1 \leq k, l \leq 4$.

In the images presented here, projections of the five connected components can be seen clearly. Each is a smooth simple closed curve. Some project

Figure 8: Projection to the 3-4 plane.



to ellipses (or even circles). The ellipses are flat, which can be seen since in certain projections the images are straight line segments. Other components are definitely non-planar.

The collection presented here includes about 250,000 points. Optically one cannot see any significant differences in comparison to the independently derived set of only 100 thousand points, except for a slight improvement in quality.

Some points in the curves escape the region chosen to be the source of starting points for the procedure. For example, the maximum of the first coordinate is 3.83. This exceeds the maximal coordinate of the starting points, which is 3.0. This phenomenon is possible because a gradient line starting in some region may evolve and converge even to a distant limit. Nevertheless, the result is that the limit points are sparse in some parts of limit curves. It is advisable to repeat the calculations with particular emphasis on these areas.

10 Comparison with O'Neil's results

The above calculations regarding the collection of self-similar systems for the Novikov circulation should be compared with the following results of O'Neil [7]. O'Neil proved that collapsing n -systems exist for any n . The basis here is Theorem 7.1.1 from p. 411 of his work:

Theorem 10.1. *Let $\kappa = (\kappa_1 \dots \kappa_n)$ be a circulation with one negative entry ($\kappa_n < 0$), and all remaining entries positive: $\kappa_l > 0$ dla $l = 1, \dots, n-1$. Let us assume that $L = \sum_{l < j} k_l k_j = 0$. By s denote the number of pairs (l, n) such that $k_l + k_n > 0$. Then the number of collinear rotating configurations is at least $s(n-2)!$.*

Of course, it is easy to find k_l satisfying these conditions, with $s > 0$. Thus, in this case, collinear rotating configurations do exist.

The existence of collinear rotating configurations implies the existence of collapsing configurations:

Theorem 10.2. *(O’Neil [7, Theorem 7.4.1, p. 413]) Let $n > 3$. For any circulation*

$$\kappa \in K_L^n = \{k = (k_1, \dots, k_n) \in R^n : k_1 = 1, \sum k_l k_j = 0\}$$

apart from an algebraic subvariety of codimension one, each rotating collinear configuration lies in a one-parameter family of collapsing configurations. Each such family is a smooth submanifold except at finitely many points.

It should be noted that the Novikov circulation is of the form $(1, \kappa, \kappa, 1)$, thus it does not satisfy the assumptions of the first O’Neil Theorem. Therefore, the existence of the Novikov examples is not a consequence of this assertion. However, some points in the Novikov circle are collinear rotating configurations. Therefore the existence of the Novikov circle is, in some sense, a consequence of the second O’Neil Theorem. On the basis of numerical calculations one can claim that two of the other four components of the self-similar configuration set, exhibited in this work, contain no collinear rotating configurations. Thus, in a sense, their existence is predicted by neither the first, nor the second O’Neil Theorem.

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